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A Class of Extremal Functions and Trigonometric Polynomials

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In this paper we present two classes of extremal approximating functions. These functions have the property that they are entire, have finite exponential type, and provide excellent approximations along the real line for a specific set of functions. One class of functions provides majorants and minorants, while the other class minimizes the L^1-norm on the real line. As applications we construct extremal trigonometric polynomials and obtain an inequality involving almost periodic trigonometric polynomials.

1. INTRODUCTION

The work presented here represents generalizations of results and techniques developed by Vaaler in [11]. Throughout this paper, the definitions for Fourier transforms and series follow those of Stein and Weiss [10]. We say that an entire function $F(z)$ has *exponential type* $\sigma > 0$ if for all $\varepsilon > 0$ there exists a constant A_ε such that

$$|F(z)| \leqslant A_\varepsilon e^{(\sigma + \varepsilon)|z|}$$

for all complex z. Finally, we write $\varepsilon(\theta) = e^{2\pi i\theta}$.

The results in [11] have their origins in a problem first considered by Beurling. He showed that the entire function

$$B(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ z^{-2} + \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)(z-n)^{-2} + 2z^{-1} \right\} \tag{1.1}$$

has several interesting properties. Specifically,

$$\operatorname{sgn}(x) \leqslant B(x), \tag{1.2}$$

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for all real x, and

$$\int_{-\infty}^{\infty} B(x) - \operatorname{sgn}(x)\, dx = 1. \tag{1.3}$$

The function $B(z)$ has exponential type 2π, and Beurling showed that it is extremal in the following sense: If $F(z)$ is any entire function of exponential type 2π which satisfies $\operatorname{sgn}(x) \leqslant F(x)$ for all real x, then

$$\int_{-\infty}^{\infty} F(x) - \operatorname{sgn}(x)\, dx \geqslant 1, \tag{1.4}$$

with equality holding in (1.4) if and only if $F(z) = B(z)$. Although Beurling never published his results, an account can be found in several places, including [1, 6, 11]. In particular, in 1974 Selberg independently discovered the function $B(z)$ and used it to obtain a sharp form of the large sieve inequality (see [8, pp. 213–226]). The more general question of approximating a given function by an entire function of finite exponential type, along with the applications of such approximations, has been studied in a number of places, among them [2–5].

We begin by defining a generalization of the function $\operatorname{sgn}(x)$. For $\alpha > 0$, let

$$R_\alpha(x) = \begin{cases} -1 & \text{if } \; x < -\alpha, \\ 1 + 2\alpha^{-1}x & \text{if } \; -\alpha \leqslant x < 0, \\ 1 & \text{if } \; 0 \leqslant x. \end{cases} \tag{1.5}$$

Letting $\alpha \to 0$, we see that $R_\alpha(x) \to \operatorname{sgn}(x)$ pointwise for all $x \neq 0$. We would like to duplicate Beurling's result with the function $\operatorname{sgn}(x)$ replaced by $R_\alpha(x)$. That is, find an entire function $B_\alpha(z)$ which has exponential type 2π, satisfies $R_\alpha(x) \leqslant B_\alpha(x)$ for all real x, and minimizes the integral along the real line of the difference $B_\alpha(x) - R_\alpha(x)$. In general this seems to be a difficult problem to solve. However, if we set $\alpha = N + 1/2$, where N is a non-negative integer, then we can find such a function. The construction of $B_\alpha(z)$ is suggested by an interpolation formula for functions $F(z)$ which are entire of exponential type 2π and are in $L^p(\mathbf{R})$ for some finite p. Provided that $F(z)$ satisfies these conditions, we have (see [11, Theorem 9])

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} F(n)(z-n)^{-2} + \sum_{m=-\infty}^{\infty} F'(m)(z-m)^{-1} \right\}. \tag{1.6}$$

Although $R_\alpha(x)$ is not an element of $L^p(\mathbf{R})$, we may use (1.6) as a guide for constructing $B_\alpha(z)$. For $\alpha = N + 1/2$, define

$$B_\alpha(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} R_\alpha(n)(z-n)^{-2} + \sum_{n=1}^{N} R'_\alpha(-n)(z+n)^{-1} + \alpha^{-1} z^{-1} \right\}. \tag{1.7}$$

If $N = 0$, then set $B_\alpha(z) = B(z)$ given in (1.1). It is clear from (1.5)–(1.7) that $B_\alpha = R_\alpha(n)$ for each integer n and that $B'_\alpha(m) = R'_\alpha(m)$ for each integer $m \neq 0$, so that it is reasonable to expect that $B_\alpha(x)$ provides a good approximation to $R_\alpha(x)$. The selection of $B'_\alpha(0) = \alpha^{-1}$ allows $B_\alpha(z)$ to satisfy the desired extremal properties.

THEOREM 1. *For each $\alpha = N + 1/2$ the function $B_\alpha(z)$ satisfies*

$$R_\alpha(x) \leqslant B_\alpha(x) \tag{1.8}$$

for all real x, and

$$\int_{-\infty}^{\infty} B_\alpha(x) - R_\alpha(x)\, dx = (4\alpha)^{-1}. \tag{1.9}$$

Furthermore, if $F(z)$ is any entire function of exponential type 2π such that $R_\alpha(x) \leqslant F(x)$ for all real x, then

$$\int_{-\infty}^{\infty} F(x) - R_\alpha(x)\, dx \geqslant (4\alpha)^{-1}, \tag{1.10}$$

with equality holding in (1.10) *if and only if $F(z) = B_\alpha(z)$.*

It is a simple matter to modify the above theorem to produce an entire function of exponential type 2π which minorizes $R_\alpha(x)$. If we let

$$b_\alpha(z) = -B_\alpha(-z-\alpha), \tag{1.11}$$

then $b_\alpha(z)$ is entire of exponential type 2π. After noting that $R_\alpha(x) = -R_\alpha(-x-\alpha)$, then by (1.8) and (1.9) it follows that

$$b_\alpha(x) \leqslant R_\alpha(x) \tag{1.12}$$

for all real x, and

$$\int_{-\infty}^{\infty} R_\alpha(x) - b_\alpha(x)\, dx = (4\alpha)^{-1}. \tag{1.13}$$

Finally, it is clear that Theorem 1 also implies that $b_\alpha(z)$ is the extremal minorizing function of exponential type 2π.

We next consider a different type of approximation problem. Before proceeding, it will be convenient to define a new function which is closely related to $R_\alpha(x)$. For each $\alpha > 0$ let

$$S_\alpha(x) = \begin{cases} \alpha^{-1} x & \text{if } |x| < \alpha, \\ \operatorname{sgn}(x) & \text{if } |x| \geqslant \alpha. \end{cases} \tag{1.14}$$

It is easily verified that $S_\alpha(x+\alpha) = R_{2\alpha}(x)$. Rather than constructing a function which majorizes or minorizes $S_\alpha(x)$, we seek an entire function of finite exponential type for which the integral

$$\int_{-\infty}^{\infty} |S_\alpha(x) - F(x)|\, dx$$

is minimized. As in the previous case, this approximation problem is difficult to solve in general. We can succeed by again setting $\alpha = N + 1/2$ and searching among functions $F(z)$ which are entire of exponential type π. If $F(z)$ is such a function and is bounded along the real line, then $F(z)$ may be expressed by the interpolation formula (see [13, Vol. II, p. 275])

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)\left\{F(0)\, z^{-1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n F(n)((z-n)^{-1} + n^{-1}) + F'(0)\right\}. \tag{1.15}$$

In a manner similar to the previous problem, we use (1.15) as a guide for constructing a function $G_\alpha(z)$ which approximates $S_\alpha(x)$. Define

$$G_\alpha(z) = \left(\frac{\sin \pi z}{\pi}\right)\left\{\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n S_\alpha(n)((z-n)^{-1} + n^{-1}) + G'_\alpha(0)\right\}, \tag{1.16}$$

where

$$G'_\alpha(0) = 2\alpha^{-1} A(N) - 2 \sum_{n=N+1}^{\infty} (-1)^n \frac{1}{n}, \tag{1.17}$$

and $A(N) = 1$ for N odd and $A(N) = 0$ for N even.

THEOREM 2. *For each $\alpha = N + 1/2$, the function $G_\alpha(z)$ satisfies*

$$0 \leqslant (-1)^N \operatorname{sgn}(\sin \pi x)\{S_\alpha(x) - G_\alpha(x)\} \tag{1.18}$$

for all real x, and

$$\int_{-\infty}^{\infty} |S_\alpha(x) - G_\alpha(x)| \, dx = (4\alpha)^{-1}. \tag{1.19}$$

Furthermore, if $F(z)$ is any entire function of exponential type π, then

$$\int_{-\infty}^{\infty} |S_\alpha(x) - F(x)| \, dx \geqslant (4\alpha)^{-1}, \tag{1.20}$$

with equality holding in (1.20) if and only if $F(z) = G_\alpha(z)$.

An analogous result, with $\mathrm{sgn}(x)$ in place of $S_\alpha(x)$, is given in [11, Theorem 4]. The proofs of Theorems 1 and 2 are contained in Section 2.

There are a variety of applications for the functions $B_\alpha(z)$ and $G_\alpha(z)$. We shall present two here. The first involves the construction of classes of extremal trigonometric polynomials. Let

$$U(x) = \max\{0, 1 - 2\,|x|\}, \tag{1.21}$$

and then define the periodic function

$$u(x) = \sum_{n=-\infty}^{\infty} U(x - n). \tag{1.22}$$

Thus $u(x)$ is the familiar triangular wave function. We use $B_\alpha(z)$ and the Poisson summation formula together with the Paley–Wiener theorem to construct trigonometric polynomials which majorize $u(x)$ and are extremal in the sense described below.

THEOREM 3. *For each nonnegative integer N there exists a trigonometric polynomial $m_N(x)$ of degree $2N$ such that*

$$u(x) \leqslant m_N(x) \tag{1.23}$$

for all real x, and

$$\int_{-1/2}^{1/2} m_N(x) - u(x) \, dx = 2(4N+2)^{-2}. \tag{1.24}$$

Furthermore, if $f(x)$ is any trigonometric polynomial of degree $2N$ or less and $u(x) \leqslant f(x)$ for all x, then

$$\int_{-1/2}^{1/2} f(x) - u(x) \, dx \geqslant 2(4N+2)^{-2}, \tag{1.25}$$

with equality holding in (1.25) if and only if $f(x) = m_N(x)$.

An easy consequence of Theorem 3 is that if we define the function

$$l_N(x) = 1 - m_N(x - 1/2),$$

then $l_N(x)$ is a trigonometric polynomial of degree $2N$ and satisfies

$$l_N(x) \leqslant u(x) \tag{1.26}$$

for all real x, and

$$\int_{-1/2}^{1/2} u(x) - l_N(x)\, dx = 2(4N+2)^{-2}. \tag{1.27}$$

It is also clear that $l_N(x)$ is the extremal minorizing trigonometric polynomial of degree $2N$. The next theorem is proved in a manner similar to Theorem 3, using the function $G_\alpha(z)$ in place of $B_\alpha(z)$.

THEOREM 4. *For each nonnegative integer N there exists a trigonometric polynomial $p_N(x)$ of degree $2N$ that satisfies*

$$0 \leqslant \operatorname{sgn}(\cos(4N+2)\,\pi x)\{u(x) - p_N(x)\}, \tag{1.28}$$

for all real x, and

$$\int_{-1/2}^{1/2} |u(x) - p_N(x)|\, dx = (4N+2)^{-2}. \tag{1.29}$$

Furthermore, if $f(x)$ is any trigonometric polynomial of degree $2N$ or less, then

$$\int_{-1/2}^{1/2} |u(x) - f(x)|\, dx \geqslant (4N+2)^{-2}, \tag{1.30}$$

with equality holding in (1.30) *if and only if $f(x) = p_N(x)$.*

The proofs of Theorems 3 and 4, along with explicit formulas for computing the coefficients $\hat{m}_N(n)$ and $\hat{p}_N(n)$, are given in Section 3.

The definitions of $B_\alpha(z)$ and $G_\alpha(z)$ may be extended to any $\alpha > 0$. For each such α, define $\Delta(\alpha)$ by

$$\Delta(\alpha) = \begin{cases} 1 - \alpha & \text{if} \quad 0 < \alpha \leqslant 1/2, \\ (4\alpha)^{-1} \left(\dfrac{\alpha}{[\alpha + 1/2] - 1/2} \right)^2 & \text{if} \quad \alpha > 1/2. \end{cases} \tag{1.31}$$

Here $[x]$ denotes the greatest integer less than or equal to x. The following are corollaries to Theorems 1 and 2, respectively.

COROLLARY 1. *For each* $\alpha > 0$, *there exists an entire function* $B_\alpha(z)$ *of exponential type* 2π *such that*

$$R_\alpha(x) \leqslant B_\alpha(x) \tag{1.32}$$

for all real x, *and*

$$\int_{-\infty}^{\infty} B_\alpha(x) - R_\alpha(x)\, dx = \Delta(\alpha). \tag{1.33}$$

COROLLARY 2. *For each* $\alpha > 0$, *there exists an entire function* $G_\alpha(z)$ *of exponential type* π *such that*

$$\int_{-\infty}^{\infty} |S_\alpha(x) - G_\alpha(x)|\, dx = \Delta(\alpha). \tag{1.34}$$

The proof of Corollary 1 is given in Section 4. The proof of Corollary 2, which is similar, is omitted. In the course of the proof, it is evident that $B_\alpha(z)$ and $G_\alpha(z)$ are extremal for $0 < \alpha < 1/2$. For $\alpha \geqslant 1/2$, the functions typically are not extremal. However, by a simple modification of the proof it can be shown that the integrals in (1.33) and (1.34) cannot be made smaller than

$$(4\alpha)^{-1} \left(\frac{\alpha}{[\alpha + 1/2] + 1/2} \right)^2,$$

subject to the constraints imposed. Thus $B_\alpha(z)$ and $R_\alpha(z)$ provide close to optimal approximations in this case.

We now consider a second application which makes use of the corollaries. Let $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers and $c_1, c_2, ..., c_N$ be arbitrary complex numbers. Define the almost periodic trigonometric polynomial

$$T(x) = \sum_{n=1}^{N} c_n e(\lambda_n x), \tag{1.35}$$

where we recall that $e(\theta) = e^{2\pi i \theta}$.

THEOREM 5. *Let* r *and* s *be real numbers, with* $s > 0$. *If* $|\lambda_n| \geqslant \varepsilon > 0$ *for each* $n = 1, 2, ..., N$, *then*

$$\left| s^{-1} \int_r^{r+s} T(x)\, dx \right| \leqslant (4\varepsilon)^{-1} \Delta(\varepsilon s) \sup_t |T'(x)|. \tag{1.36}$$

If $T(x)$ happens to be a real-valued function, then

$$\left| s^{-1} \int_r^{r+s} T(x)\,dx \right| \leqslant (2\varepsilon)^{-1} \Delta(\varepsilon s) \sup_t T'(t). \tag{1.37}$$

Letting $s \to 0$ in (1.36) and (1.37), we recover inequalities first proved by Bohr [9, p. 142] and Beurling [11, Theorem 15], respectively. The latter inequality was Beurling's motivation for constructing the function $B(z)$. The proof of Theorem 5 is given in Section 4.

2. PROOF OF THEOREMS 1 AND 2

In order to simplify some later expressions, we define

$$C(x) = \sum_{n=1}^{\infty} (x+n)^{-2}. \tag{2.1}$$

We pause to note the simple identity

$$\left(\frac{\sin \pi x}{\pi}\right)^2 \sum_{n=1}^{N} (x+n)^{-2} = \left(\frac{\sin \pi x}{\pi}\right)^2 \{C(x) - C(x+N)\}, \tag{2.2}$$

which will be used in the work that follows. We shall also require the bounds for $C(x)$ which are given below.

LEMMA 1. *If $x > -1/2$, then*

$$C(x) < \frac{2}{2x+1}, \tag{2.3}$$

and if $x > 0$, then

$$\frac{1}{x} - \frac{1}{2x^2} < C(x). \tag{2.4}$$

Proof. Assume $x > -1/2$. Since t^{-2} is concave up for all $t > 0$, we have

$$C(x) < \sum_{n=1}^{\infty} \int_{x+n-1/2}^{x+n+1/2} t^{-2}\,dt$$

$$= \int_{x+1/2}^{\infty} t^{-2}\,dt = \frac{2}{2x+1}.$$

This verifies (2.3). If $x>0$, then upon applying the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\frac{1}{x} &= \int_x^\infty t^{-2}\,dt\\ &= \sum_{n=0}^\infty \int_{x+n}^{x+n+1} t^{-2}\,dt\\ &= \sum_{n=0}^\infty (x+n)^{-1}(x+n+1)^{-1}\\ &< \sum_{n=0}^\infty \frac{1}{2}\{(x+n)^{-2}+(x+n+1)^{-2}\} = \frac{1}{2x^2}+C(x).\end{aligned}$$

Thus (2.4) now follows.

Proof of Theorem 1. First suppose that $\alpha = 1/2$, so that $B_\alpha(z) = B(z)$. If $x \leqslant -1/2$ or $0 \leqslant x$, then $R_\alpha(x) = \operatorname{sgn}(x)$, so that (1.8) follows from (1.2). Thus we may assume that $-1/2 < x < 0$, in which case $R_\alpha(x) = 1+4x$. Recalling the identity

$$1 = \left(\frac{\sin \pi x}{\pi}\right)^2 \sum_{n=-\infty}^{\infty} (x-n)^{-2} = \left(\frac{\sin \pi x}{\pi}\right)^2 \{x^{-2} + C(x) + C(-x)\}, \tag{2.5}$$

we may express $R_\alpha(x)$ and $B_\alpha(x)$ by

$$R_\alpha(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \{x^{-2} + 4x^{-1} + (1+4x)(C(x)+C(-x))\}, \tag{2.6}$$

$$B_\alpha(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \{x^{-2} + 2x^{-1} + C(-x) - C(x)\}. \tag{2.7}$$

Subtracting (2.6) from (2.7) and applying Lemma 1 yields

$$\begin{aligned}B_\alpha(x) - R_\alpha(x) &= \left(\frac{\sin \pi x}{\pi}\right)^2 \{-2x^{-1} - 4xC(-x) - (4x+2)\,C(x)\}\\ &\geqslant \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{-2x^{-1} - 4x\left(\frac{1}{-x} - \frac{1}{2x^2}\right) - (4x+2)\left(\frac{2}{2x+1}\right)\right\}\\ &= 0.\end{aligned}$$

Therefore (1.8) holds for all real x. We also have, by (1.3),

$$\int_{-\infty}^{\infty} B_\alpha(x) - R_\alpha(x)\, dx = \int_{-\infty}^{\infty} B(x) - \operatorname{sgn}(x)\, dx - \int_{-\infty}^{\infty} R_\alpha(x) - \operatorname{sgn}(x)\, dx$$
$$= 1 - 1/2 = 1/2,$$

which verifies (1.9). The inequality (1.10) and the uniqueness of $B_\alpha(z)$ follows from (1.4) and the uniqueness of $B(z)$.

We now assume that $N \geqslant 1$. To establish (1.8) we consider three cases corresponding to the three parts of the piece-wise defined $R_\alpha(x)$. In the course of verifying (1.8), we shall also show that $B_\alpha(x) - R_\alpha(x)$ is integrable, which will be required to evaluate the integral in (1.9). Before proceeding, we first establish an identity which will be used in all three cases. For each integer n, replace $R_\alpha(n)$ and $R'_\alpha(n)$ in (1.7) with the function values from (1.5). Applying the identity (2.5), we have

$$B_\alpha(x) - 1 = \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ -C(x) - C(x+N) + (\alpha x)^{-1} + (2\alpha)^{-1} \times \sum_{n=1}^{N} \left(\frac{2\alpha - 4n}{(x+n)^2} + \frac{4}{x+n} \right) \right\}.$$

Placing the expression in the sum above over a common denominator, applying (2.2) and simplifying produces

$$B_\alpha(x) - 1 = \alpha^{-1} \left(\frac{\sin \pi x}{\pi}\right)^2 \{2xC(x) - 2(\alpha + x)\, C(x+N) + x^{-1}\}. \tag{2.8}$$

Now suppose that $x \geqslant 0$, so that in this case $R_\alpha(x) = 1$. If $x = 0$, then $B_\alpha(x) = R_\alpha(x)$, so that we may assume that $x > 0$. Since $B_\alpha(x) - R_\alpha(x) = B_\alpha(x) - 1$, we may apply Lemma 1 to the term in brackets on the right of (2.8) to obtain the lower bound

$$2xC(x) - 2(\alpha + x)\, C(x+N) + x^{-1} > 2x\left(\frac{1}{x} - \frac{1}{2x^2}\right) - 2(\alpha + x) \times \left(\frac{2}{2(x+N)+1}\right) + x^{-1} = 0.$$

Thus it follows that $R_\alpha(x) \leqslant B_\alpha(x)$ for all $x \geqslant 0$. We may also apply Lemma 1 to (2.8) to obtain an upper bound for $B_\alpha(x) - R_\alpha(x)$. After simplifying we find that

$$B_\alpha(x) - R_\alpha(x) \leqslant \alpha^{-1} \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \frac{1}{x(2x+1)} + \frac{1}{2(x+N)^2} \right\}.$$

It follows that $B_\alpha(x)-R_\alpha(x)$ is integrable on the positive real line. Next suppose that $-\alpha<x<0$. In this case $R_\alpha(x)=1+2\alpha^{-1}x$, so that $B_\alpha(x)-R_\alpha(x)=(B_\alpha(x)-1)-2\alpha^{-1}x$. By (2.5) we have

$$2\alpha^{-1}x=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)^2\{2x^{-1}+2xC(x)+2xC(-x)\},$$

so that together with (2.8) and an application of Lemma 1, it follows that

$$B_\alpha(x)-R_\alpha(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)^2\{-x^{-1}-2xC(-x)-2(\alpha+x)\,C(x+N)\}\geqslant 0.$$

Finally, suppose that $x<-\alpha$, so that $R_\alpha(x)=-1$. Writing $B_\alpha(x)-R_\alpha(x)=(B_\alpha(x)-1)+2$ and applying (2.5) and (2.8) yields

$$B_\alpha(x)-R_\alpha(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)^2\{2\alpha(x^{-2}+C(-x))+2(\alpha+x)\\ \times(C(x)-C(x+N))+x^{-1}\}.$$

Using the identity

$$\left(\frac{\sin \pi x}{\pi}\right)^2\{C(x)-C(x+N)\}\\ =\left(\frac{\sin \pi x}{\pi}\right)^2\{C(-x-N-1)-C(-x)-x^{-2}\}$$

and applying Lemma 1 yields

$$B_\alpha(x)-R_\alpha(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)^2\\ \times\{-x^{-1}-2xC(-x)+2(\alpha+x)\,C(-x-N-1)\}\geqslant 0.$$

This settles (1.8) for all $x<-\alpha$. The inequality $R_\alpha(x)\leqslant B_\alpha(x)$ for $x=-\alpha$ follows from the continuity of $R_\alpha(x)$ and $B_\alpha(x)$, so that (1.8) holds for all real x. Using the preceeding representation of $B_\alpha(x)-R_\alpha(x)$ for $x<-\alpha$ and applying Lemma 1 a final time, we obtain the upper bound

$$B_\alpha(x)-R_\alpha(x)\leqslant\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)^2\left\{\frac{1}{2(x+N+1)^2}+\frac{1}{x(2x-1)}\right\}.$$

Combining this with our earlier work implies that $B_\alpha(x)-R_\alpha(x)$ is integrable along the entire real line. We next establish the lower bound given in (1.10), and in process will also show that (1.9) holds. Suppose that $F(z)$ is an entire function of exponential type 2π such that $R_\alpha(x) \leqslant F(x)$ for all real x. We may also assume without any loss that

$$\int_{-\infty}^{\infty} F(x) - R_\alpha(x)\, dx < \infty.$$

Now define the functions

$$M(z) = \tfrac{1}{2}\{F(z) + F(-z)\} \tag{2.9}$$

and

$$L_\alpha(x) = \tfrac{1}{2}\{R_\alpha(x) + R_\alpha(-x)\}. \tag{2.10}$$

It follows easily from (2.9) and (2.10) that

$$\int_{-\infty}^{\infty} F(x) - R_\alpha(x)\, dx = \int_{-\infty}^{\infty} M(x) - L_\alpha(x)\, dx. \tag{2.11}$$

Next we note that

$$L_\alpha(x) = \max\{0, 1 - \alpha^{-1}\, |x|\},$$

so that clearly $L_\alpha(x)$ is integrable, with

$$\int_{-\infty}^{\infty} L_\alpha(x)\, dx = \alpha. \tag{2.12}$$

Thus it follows from (2.11) that $M(x)$ is also integrable. From the definition (2.9) it is clear that $M(z)$ is an entire function of exponential type 2π, so that by the Poisson summation formula ([12], pp. 104–105) we find that

$$\begin{aligned} \int_{-\infty}^{\infty} M(x)\, dx &= \sum_{n=-\infty}^{\infty} M(n) \\ &\geqslant \sum_{n=-\infty}^{\infty} L_\alpha(n) \\ &= \sum_{n=-N}^{N} (1 - \alpha^{-1}\, |n|) \\ &= \alpha + (4\alpha)^{-1}. \end{aligned} \tag{2.13}$$

Combining together (2.11), (2.12) and (2.13) now yields (1.10). We recall that $B_\alpha(n)=R_\alpha(n)$ for each integer n, so that if $F(z)=B_\alpha(z)$ then the inequality in (2.13) holds with equality. This in turn implies that (1.10) holds with equality, which verifies (1.9). By the same reasoning, in order for equality to hold in (1.10) it must be that $F(n)=R_\alpha(n)$ for all integers n. Moreover, as $R_\alpha(x)\leqslant F(x)$ for all real x, it follows that $F'(n)=R'_\alpha(n)$ for each integer $n\neq 0$. The function $B_\alpha(z)$ also satisfies these conditions, and the difference $F(z)-B_\alpha(z)$ is an integrable entire function of exponential type 2π. Applying the interpolation formula (1.6) and the above remarks, we find that

$$F(z)-B_\alpha(z)=c\left(\frac{\sin \pi z}{\pi}\right)^2\frac{1}{z} \tag{2.14}$$

for some constant c. The right side of (2.14) is integrable only if $c=0$, which implies that $F(z)=B_\alpha(z)$. Thus $B_\alpha(z)$ is the unique extreme majorant, and the proof is complete.

We take a few moments to establish some preliminary results that will be required in the proof of Theorem 2. As a notational convenience, for all $x>-1$ define

$$D(x)=\sum_{n=1}^{\infty}(x+2n)^{-1}(x+2n-1)^{-1}.$$

LEMMA 2. *If* $-1/2<x$, *then*

$$\frac{1}{x+1}-\frac{1}{2x-3}<D(x)<\frac{1}{2x+1}. \tag{2.15}$$

Proof. By the arithmetic-geometric mean inequality and Lemma 1, we have

$$\begin{aligned} D(x)&=\sum_{n=1}^{\infty}(x+2n)^{-1}(x+2n-1)^{-1}\\ &<\frac{1}{2}\sum_{n=1}^{\infty}\{(x+2n)^{-2}+(x+2n-1)^{-2}\}\\ &=\frac{1}{2}C(x)<\frac{1}{2x+1}. \end{aligned}$$

This verifies the upper bound in (2.15). To obtain the lower bound, we note that

$$D(x)+D(x+1)=\lim_{M\to\infty}\sum_{n=1}^{M}\{(x+2n)^{-1}(x+2n-1)^{-1}$$
$$+(x+2n)^{-1}(x+2n+1)^{-1}\}$$

$$=\lim_{M\to\infty}\sum_{n=1}^{M}\{(x+2n-1)^{-1}-(x+2n+1)^{-1}\}$$
$$=\lim_{M\to\infty}\{(x+1)^{-1}-(x+2M+1)^{-1}\}=(x+1)^{-1}.$$

Thus it follows that

$$D(x)=(x+1)^{-1}-D(x+1),$$

and applying the upper bound to $D(x+1)$ on the right above completes the proof.

By using the same ideas as in the proof above, we may verify for all $x>-1$ that

$$\sum_{n=1}^{N}(-1)^n(x+n)^{-1}=(-1)^N D(x+N)-D(x), \tag{2.16}$$

an identity which shall needed in the proof of Theorem 2. It may be shown (see [11], Theorem 4) that the function

$$G(z)=\left(\frac{\sin \pi z}{\pi}\right)\left\{\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{sgn}(n)((z-n)^{-1}+n^{-1})+\log 4\right\}$$

is entire of exponential type π, that

$$\int_{-\infty}^{\infty}|\operatorname{sgn}(x)-G(x)|\,dx=1,$$

and that $G'(x)$ is integrable. Suppose that $F(z)$ is an entire function of exponential type π such that

$$\int_{-\infty}^{\infty}|S_\alpha(x)-F(x)|\,dx<\infty.$$

As the differences $S_\alpha(x)-\operatorname{sgn}(x)$ and $\operatorname{sgn}(x)-G(x)$ are also integrable, it follows from the triangle inequality that $F(x)-G(x)$ is integrable. The difference $F(z)-G(z)$ is entire of exponential type π, so that by a result of Plancherel and Polya [7] it follows that $F'(x)-G'(x)$ is integrable, and thus $F'(x)$ is integrable. Now define

$$E_\alpha(x)=S_\alpha(x)-F(x), \qquad H(z)=F'(z).$$

LEMMA 3. *For each $\alpha>0$, the Fourier transform of $E_\alpha(x)$ is given by*

$$\hat{E}_\alpha(t)=\frac{1}{2\pi it}\left\{\frac{\sin 2\alpha\pi t}{\alpha\pi t}-\hat{H}(t)\right\}. \tag{2.17}$$

Proof. By integrating directly we have

$$\int_{-\infty}^{\infty} e(-tx)\,dE_\alpha(x)=\alpha^{-1}\int_{-\alpha}^{\alpha} e(-tx)\,dx-\int_{-\infty}^{\infty} F'(x)\,e(-tx)\,dx$$
$$=\frac{\sin 2\alpha\pi t}{\alpha\pi t}-\hat{H}(t). \tag{2.18}$$

Integrating by parts on the left side of (2.18) and solving for $\hat{E}_\alpha(t)$ then yields (2.17).

Proof of Theorem 2. To establish (1.18), we first note that if $x=0$ then the result holds trivially. Next, as $\operatorname{sgn}(\sin\pi x)\{S_\alpha(x)-G_\alpha(x)\}$ is an even function, it suffices to assume that $x>0$. Let $M>N$ be an integer, and define

$$G_{\alpha,M}(z)=\alpha^{-1}\left(\frac{\sin\pi z}{\pi}\right)\Bigg\{2A(N)+\sum_{\substack{n=-N\\ n\neq 0}}^{N} n(-1)^n\left((z-n)^{-1}+n^{-1}\right)$$
$$+\alpha\sum_{N+1\leqslant|n|\leqslant M}(-1)^n\operatorname{sgn}(n)\left((z-n)^{-1}+n^{-1}\right)$$
$$-2\alpha\sum_{n=N+1}^{M}(-1)^n n^{-1}\Bigg\}.$$

It follows from (1.14), (1.16) and (1.17) that $G_{\alpha,M}(z)\to G_\alpha(z)$ uniformly on compact subsets of $\mathbf{C}$ as $M\to\infty$. Simplifying above, we have

$$G_{\alpha,M}(z)=\alpha^{-1}\left(\frac{\sin\pi z}{\pi}\right)\Bigg\{\sum_{\substack{n=-N\\ n\neq 0}}^{N}(-1)^n n(z-n)^{-1}$$
$$+\alpha\sum_{N+1\leqslant|n|\leqslant M}(-1)^n\operatorname{sgn}(n)(z-n)^{-1}\Bigg\}. \tag{2.19}$$

As in the proof of Theorem 1, we now consider different cases. First, suppose that $0<x<\alpha$, so that $S_\alpha(x)=\alpha^{-1}x$. Define

$$S_{\alpha,M}(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)\left\{1+x\sum_{\substack{n=-M\\ n\neq 0}}^{M}(-1)^n(x-n)^{-1}\right\}, \tag{2.20}$$

so that $S_{\alpha,M}(x)\to S_\alpha(x)$ as $M\to\infty$. Subtracting (2.19) from (2.20), combining sums and simplifying, we find that

$$\begin{aligned}S_{\alpha,M}(x)-G_{\alpha,M}(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)\Bigg\{&1-2A(N)+(x-\alpha)\\ &\times\sum_{n=N+1}^{M}(-1)^n(x-n)^{-1}\\ &+(x+\alpha)\sum_{n=N+1}^{M}(-1)^n(x+n)^{-1}\Bigg\}.\end{aligned}$$

Reindexing the sums and using (2.16), it follows that the right side of the expression above is equal to

$$\begin{aligned}&\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)\{1-2A(N)+(-1)^N(\alpha-x)((-1)^{M-N}D(-x+M)\\ &\quad -D(-x+N))+(-1)^N(x+\alpha)((-1)^{M-N}D(x+M)-D(x+N))\}.\end{aligned}$$

Letting $M\to\infty$ yields

$$\begin{aligned}S_\alpha(x)-G_\alpha(x)=\alpha^{-1}\left(\frac{\sin \pi x}{\pi}\right)&\{1-2A(N)+(-1)^N(x-\alpha)\,D(-x+N)\\ &-(-1)^N(x+\alpha)\,D(x+N)\}.\end{aligned}$$

Taking N to be even and applying Lemma 2 to the expression in brackets above, we have

$$\begin{aligned}&1+(x-\alpha)\,D(-x+N)-(x+\alpha)\,D(x+N)\\ &\quad\geqslant 1+(x-\alpha)(2(N-x)+1)^{-1}-(x+\alpha)(2(N+x)+1)^{-1},\end{aligned}$$

and as the right side above is zero, it follows that (1.18) holds for $0<x<\alpha$. The case for N odd is similar. Now suppose that $x>\alpha$, so that $S_\alpha(x)=1$. Here we define

$$S_{\alpha,M}(x)=\left(\frac{\sin \pi x}{\pi}\right)\left\{x^{-1}+\sum_{\substack{n=-M\\ n\neq 0}}^{M}(-1)^n(x-n)^{-1}\right\}, \tag{2.21}$$

so that again $S_{\alpha, M}(x) \to S_\alpha(x)$ as $M \to \infty$. Subtracting (2.19) from (2.21) and combining sums, we have

$$S_{\alpha, M}(x) - G_{\alpha, M}(x) = \alpha^{-1} \left(\frac{\sin \pi x}{\pi}\right) \left\{ \alpha x^{-1} + \sum_{\substack{n=-N \\ n \neq 0}}^{N} (-1)^n (\alpha - n)(x-n)^{-1} \right.$$
$$\left. + 2\alpha \sum_{n=N+1}^{M} (-1)^n (x+n)^{-1} \right\}.$$

We next add and subtract the expression

$$x \sum_{\substack{n=-N \\ n \neq 0}}^{N} (-1)^n (x-n)^{-1}$$

to the first sum on the right above. After combining terms, reindexing the resulting sums and applying (2.16) we have

$$S_{\alpha, M}(x) - G_{\alpha, M}(x) = \alpha^{-1} \left(\frac{\sin \pi x}{\pi}\right) \{1 - 2A(N) + (-1)^N (\alpha - x) D(x - N - 1)$$
$$- (-1)^N (\alpha + x) D(x+N) + 2\alpha (-1)^M D(x+M)\}.$$

Letting $M \to \infty$ then yields

$$S_\alpha(x) - G_\alpha(x) = \alpha^{-1} \left(\frac{\sin \pi x}{\pi}\right) \{1 - 2A(N) + (-1)^N (\alpha - x) D(x - N - 1)$$
$$- (-1)^N (\alpha + x) D(x+N)\}.$$

As in the preceding case, applying Lemma 2 to the expression in brackets serves to verify (1.18) for all $x > \alpha$. The case $x = \alpha$ follows from the continuity of $S_\alpha(x) - G_\alpha(x)$, so that (1.18) holds for all x. A second application of Lemma 2 produces an upper bound for $S_\alpha(x) - G_\alpha(x)$. Taking N even,

$$S_\alpha(x) - G_\alpha(x)$$
$$\leqslant \alpha^{-1} \left|\frac{\sin \pi x}{\pi}\right| \left\{ \frac{1}{4(x - \alpha + 1)(x - N)} + \frac{1}{4(x + \alpha + 1)(x + N + 1)} \right\}.$$

Thus $S_\alpha(x) - G_\alpha(x)$ is integrable along the real line. The case for N odd is similar.

We now verify the lower bound (1.20), and in the process it will be evident that (1.19) holds as well. Suppose that $F(z)$ is an entire function of exponential type π, and that $S_\alpha(x) - F(x)$ is integrable. As in Lemma 3, let $E_\alpha(x) = S_\alpha(x) - F(x)$ and $H(z) = F'(z)$, and then observe that

$$\int_{-\infty}^{\infty} |S_\alpha(x) - F(x)|\, dx \geqslant \left| \int_{-\infty}^{\infty} \operatorname{sgn}(\sin \pi x)\, E_\alpha(x)\, dx \right|. \tag{2.22}$$

The function $\operatorname{sgn}(\sin \pi x)$ has period 2 and Fourier series

$$\operatorname{sgn}(\sin \pi x) = \frac{2}{\pi i} \sum_{n=-\infty}^{\infty} (2n+1)^{-1}\, e((n+1/2)x). \tag{2.23}$$

Applying this expansion to the right side of (2.22) yields

$$\begin{aligned} &\left| \frac{2}{\pi i} \int_{-\infty}^{\infty} E_\alpha(x) \sum_{n=-\infty}^{\infty} (2n+1)^{-1}\, e((n+1/2)x)\, dx \right| \\ &\quad = \left| \frac{2}{\pi i} \sum_{n=-\infty}^{\infty} (2n+1)^{-1}\, \hat{E}_\alpha(-(n+1/2)) \right|. \end{aligned}$$

Since $F(z)$ has exponential type π, so does $F'(z)$, and hence it follows that $\hat{H}(t) = 0$ for all $|t| \geqslant 1/2$. By Lemma 3, the preceeding expression is equal to

$$\begin{aligned} \left| \frac{2}{\pi i} \sum_{n=-\infty}^{\infty} (2n+1)^{-1} \frac{\sin 2\alpha\pi(n+1/2)}{2\alpha\pi^2 i(n+1/2)^2} \right| &= \frac{4}{\alpha\pi^3} \left| \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^{-3} \right| \\ &= (4\alpha)^{-1}. \end{aligned}$$

This verifies the lower bound given in (1.20). By (1.18) it is clear that there is equality in (2.22) when $F(z) = G_\alpha(z)$. Thus in this case equality holds in (1.20), so that

$$\int_{-\infty}^{\infty} |S_\alpha(x) - G_\alpha(x)|\, dx = (4\alpha)^{-1},$$

which is (1.19). If $F(z)$ is any entire function of exponential type π for which (1.20) holds with equality, then since $S_\alpha(x) - F(x)$ is continuous it follows that $S_\alpha(n) = F(n)$ for each integer n, which in turn implies that

$$F(n) = G_\alpha(n) \tag{2.24}$$

for each integer n. Clearly $F(z)-G_\alpha(z)$ has exponential type π and is integrable along the real line. Hence the interpolation formula (1.15) is applicable, and in view of (2.24) it follows that

$$F(z)-G_\alpha(z)=c\sin\pi z \tag{2.25}$$

for some constant c. In order for the right side of (2.25) to be integrable, $c=0$, which implies that $F(z)=G_\alpha(z)$. Thus $G_\alpha(z)$ is the unique extremal function and the proof is complete.

3. PROOF OF THEOREMS 3 AND 4

We begin with a description of the construction of the trigonometric polynomial $m_N(x)$ given in Theorem 3. Let $\alpha=N+1/2$, and define

$$M_N(z)=\tfrac{1}{2}\{B_\alpha(z)+B_\alpha(-z)\}. \tag{3.1}$$

Recall the function $L_\alpha(x)$ in (2.10) given by

$$L_\alpha(x)=\tfrac{1}{2}\{R_\alpha(x)+R_\alpha(-x)\}.$$

It is clear that $L_\alpha(x)\leqslant M_N(x)$ for all real x, and by applying (2.11) with $F(z)=B_\alpha(z)$ we deduce that

$$\int_{-\infty}^{\infty} M_N(x)-L_\alpha(x)\,dx=(4N+2)^{-1}. \tag{3.2}$$

Next we note that $L_\alpha((2N+1)x)=U(x)$, where $U(x)$ is defined in (1.21). It follows that

$$U(x)\leqslant M_N((2N+1)x) \tag{3.3}$$

for all real x, and by an easy change of variables and (3.2) we have

$$\int_{-\infty}^{\infty} M_N((2N+1)x)-U(x)\,dx=2(4N+2)^{-2}. \tag{3.4}$$

The function $M_N(x)$ is integrable, with Fourier transform given by

$$(2N+1)^{-1}\,\hat{M}_N((2N+1)^{-1}\,t). \tag{3.5}$$

A formula for $\hat{M}_N(t)$ is contained in the following lemma.

LEMMA 4. *The function $M_N(z)$ has Fourier transform supported* $(-1, 1)$, *with*

$$\hat{M}_N(t) = \left\{(1-|t|)\,\frac{\sin^2(N+1)\,\pi t + \sin^2 N\pi t}{(2N+1)\sin^2 \pi t} + \frac{2\,\mathrm{sgn}(t)\sin(N+1)\,\pi t \sin N\pi t}{(2N+1)\,\pi \sin \pi t}\right\} \tag{3.6}$$

for all $|t| < 1$.

Proof. The support of $\hat{M}_N(t)$ follows immediately from the observations that $M_N(z)$ has exponential type 2π, that $\hat{M}_N(t)$ is continuous, and an application of the Paley–Wiener theorem ([10], pp. 108–114). Now define

$$K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 = \int_{-1}^{1} (1-|t|)\, e(tz)\, dt, \tag{3.7}$$

$$J(z) = zK(z) = (2\pi i)^{-1} \int_{-1}^{1} \mathrm{sgn}(t)\, e(tz)\, dt. \tag{3.8}$$

From the definition of $B_\alpha(z)$ and (3.1), we have

$$M_N(z) = \sum_{n=-N}^{N} \left\{\left(1 - \frac{2\,|n|}{2N+1}\right) K(z-n) - \frac{2}{2N+1}\,\mathrm{sgn}(n)\, J(z-n)\right\}. \tag{3.9}$$

Replacing the functions $K(z)$ and $J(z)$ in (3.9) with their integral representations from (3.7) and (3.8) and then reversing the order of the integrals and sums, $M_N(z)$ may be written

$$M_N(z) = \int_{-1}^{1} \left\{(1-|t|) \sum_{n=-N}^{N} \left(1 - \frac{2\,|n|}{2N+1}\right) e(-nt) - \frac{\mathrm{sgn}(t)}{(2N+1)\,\pi i} \sum_{n=-N}^{N} \mathrm{sgn}(n)\, e(-nt)\right\} e(tz)\, dt. \tag{3.10}$$

The expression within the brackets on the right of (3.10) is then the Fourier transform of $M_N(z)$. Applying the identities

$$\sum_{n=-N}^{N} \left(1 - \frac{2\,|n|}{2N+1}\right) e(-nt) = \frac{\sin^2(N+1)\,\pi t + \sin^2 N\pi t}{(2N+1)\sin^2 \pi t}$$

and

$$\sum_{n=-N}^{N} \operatorname{sgn}(n)\, e(-nt) = -\frac{2i \sin(N+1)\,\pi t \sin N\pi t}{\sin \pi t}$$

completes the proof.

Now define the periodic function

$$m_N(x) = \sum_{n=-\infty}^{\infty} M_N((2N+1)(x-n)). \tag{3.11}$$

By the Poisson summation formula, (3.5) and Lemma 4, it follows that

$$\begin{aligned} m_N(x) &= \sum_{n=-\infty}^{\infty} (2N+1)^{-1}\, \hat{M}_N((2N+1)^{-1}\, n)\, e(nx) \\ &= (2N+1)^{-1} \sum_{n=-2N}^{2N} \hat{M}_N((2N+1)^{-1}\, n)\, e(nx). \end{aligned} \tag{3.12}$$

Proof of Theorem 3. It is clear from (3.12) and Lemma 4 that $m_N(x)$ has degree $2N$. The inequality (1.23) follows from the definition of $u(x)$ given in (1.22), the inequality (3.3) and (3.11). The integral (1.24) may be deduced from (3.4). To establish that $m_N(x)$ is extremal, suppose that $f(x)$ is a trigonometric polynomial of degree $2N$ or less, and that $u(x) \leqslant f(x)$ for all real x. Then

$$\begin{aligned} 0 &\leqslant \sum_{m=-N}^{N} \left\{ f\left(\frac{m}{2N+1}\right) - u\left(\frac{m}{2N+1}\right) \right\} \\ &= \sum_{m=-N}^{N} \sum_{n=-2N}^{2N} \hat{f}(n)\, e\left(\frac{mn}{2N+1}\right) - \sum_{m=-N}^{N} \left(1 - 2\left|\frac{m}{2N+1}\right|\right) \\ &= \sum_{n=-2N}^{2N} \hat{f}(n) \sum_{m=-N}^{N} e\left(\frac{mn}{2N+1}\right) - \frac{1}{2}(2N+1) - (4N+2)^{-1} \\ &= (2N+1)\, \hat{f}(0) - \frac{1}{2}(2N+1) - (4N+2)^{-1}. \end{aligned}$$

Thus we have

$$\tfrac{1}{2} + 2(4N+2)^{-2} \leqslant \hat{f}(0),$$

and the lower bound (1.25) now follows. In order for equality to hold in (1.25) it must be that there is equality in the preceeding inequalities. This implies that

$$f\left(\frac{m}{2N+1}\right)=u\left(\frac{m}{2N+1}\right)$$

for each integer m satisfying $-N \leqslant m \leqslant N$. Since $u(x) \leqslant f(x)$, it also must be that

$$f'\left(\frac{m}{2N+1}\right)=u'\left(\frac{m}{2N+1}\right)$$

for each $m \neq 0$, $-N \leqslant m \leqslant N$. As $f(x)$ has degree at most $2N$, these $4N+1$ conditions completely determine the trigonometric polynomial ([13], Vol. II, p. 23), so that $f(x)$ is unique. Thus $m_N(x)$ is the unique extreme majorant of degree $2N$.

The construction of $p_N(x)$ and the proof of Theorem 4 use ideas similar to those just employed. To begin, let

$$P_N(z)=\tfrac{1}{2}\{G_\alpha(z+N+1/2)+G_\alpha(-z+N+1/2)\}, \tag{3.13}$$

where again $\alpha=N+1/2$. Suppose that

$$V_N(z)=\tfrac{1}{2}\{S_\alpha(z+N+1/2)+S_\alpha(-z+N+1/2)\}. \tag{3.14}$$

Applying (1.18) we see that

$$\begin{aligned}0 &\leqslant (-1)^N \operatorname{sgn}(\sin \pi(x+N+1/2))\{S_\alpha(x+N+1/2)-G_\alpha(x+N+1/2)\}\\ &= \operatorname{sgn}(\cos \pi x)\{S_\alpha(x+N+1/2)-G_\alpha(x+N+1/2)\},\end{aligned}$$

and likewise we have

$$0 \leqslant \operatorname{sgn}(\cos \pi x)\{S_\alpha(-x+N+1/2)-G_\alpha(-x+N+1/2)\}.$$

It plainly follows that

$$0 \leqslant \operatorname{sgn}(\cos \pi x)\{V_N(x)-P_N(x)\}. \tag{3.15}$$

Applying this inequality together with (1.19), we have

$$\begin{aligned}\int_{-\infty}^{\infty} |V_N(x)-P_N(x)|\,dx &= \int_{-\infty}^{\infty} \operatorname{sgn}(\cos \pi x)\{V_N(x)-P_N(x)\}\,dx\\ &= \int_{-\infty}^{\infty} (-1)^N \operatorname{sgn}(\sin \pi x)\{S_\alpha(x)-G_\alpha(x)\}\,dx\\ &= (4N+2)^{-1}. \end{aligned} \tag{3.16}$$

Next we note that

$$U(x) = V_N((4N+2)x),$$

which implies

$$u(x) = \sum_{n=-\infty}^{\infty} V_N((4N+2)(x-n)).$$

$P_N(z)$ is entire of exponential type π, and by (3.16) $P_N(x)$ is integrable. Hence by the Poisson summation formula and the Paley–Wiener theorem, we may define the trigonometric polynomial

$$p_n(x) = \sum_{n=-\infty}^{\infty} P_N((4N+2)(x-n))$$

$$= (4N+2)^{-1} \sum_{n=-2N}^{2N} \hat{P}_N((4N+2)^{-1}\, n)\, e(nx). \tag{3.17}$$

The coefficients $\hat{p}_N(n)$ may be explicitly determined using the following result.

Lemma 5. *The function* $P_N(z)$ *has Fourier transform supported on* $(-1/2, 1/2)$, *with*

$$\hat{P}_N(t) = \frac{\sin^2(2N+1)\,\pi t \cos \pi t}{(2N+1)\sin^2 \pi t} \tag{3.18}$$

for all $|t| < 1/2$.

Proof. First, support of $\hat{P}_N(t)$ follows from the remarks preceeding the lemma. Next, if $F(z)$ is an entire function of exponential type π and $F(x)$ is integrable, then $F(z)$ may be expressed by an interpolation formula that is somewhat simpler than the one given in (1.15). In this case we have (see [12], p. 107)

$$F(z) = \sum_{n=-\infty}^{\infty} F(n)\, I(z-n), \tag{3.19}$$

where

$$I(z) = \left(\frac{\sin \pi z}{\pi z}\right) = \int_{-1/2}^{1/2} e(tz)\, dt. \tag{3.20}$$

In what follows, it will be convenient to have a translated version of the formula in (3.19). By an easy change of variables argument, we may transform (3.19) into

$$F(z)=\sum_{n=-\infty}^{\infty} F(n+1/2)\, I(z-n-1/2). \tag{3.21}$$

For each integer n we have $P_N(n+1/2)=V_N(n+1/2)$, and since

$$V_N(x)=\max\{0, 1-(2N+1)^{-1}\,|x|\},$$

it follows from (3.20) and (3.21) that

$$\begin{aligned}
P_N(z)&=\sum_{n=-(2N+1)}^{2N}\left(1-\frac{|n+1/2|}{2N+1}\right) I(z-n-1/2)\\
&=\int_{-1/2}^{1/2}\left\{\sum_{n=-(2N+1)}^{2N}\left(1-\frac{|n+1/2|}{2N+1}\right) e(-(n+1/2)\,t)\right\} e(tz)\,dt.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\hat{P}_N(t)&=\sum_{n=-(2N+1)}^{2N}\left(1-\frac{|n+1/2|}{2N+1}\right) e(-(n+1/2)\,t)\\
&=\sum_{n=-(4N+1)}^{4N+1}\left(1-\frac{|n|}{4N+2}\right) e(nt/2)-\sum_{n=-2N}^{2N}\left(1-\frac{|n|}{2N+1}\right) e(nt)\\
&=\frac{1}{4N+2}\left(\frac{\sin(2N+1)\,\pi t}{\sin \pi t/2}\right)^2-\frac{1}{2N+1}\left(\frac{\sin(2N+1)\,\pi t}{\sin \pi t}\right)^2\\
&=\frac{\sin^2(2N+1)\,\pi t\cos \pi t}{(2N+1)\sin^2 \pi t}
\end{aligned}$$

which is (3.18), and the proof is complete.

Proof of Theorem 4. It is clear from Lemma 5 that $p_N(x)$ has degree $2N$. To verify (1.29), we apply (3.15) to obtain

$$\begin{aligned}
0&\leqslant \sum_{n=-\infty}^{\infty} \operatorname{sgn}(\cos(4N+2)\,\pi(x-n))\{V_N((4N+2)(x-n))\\
&\qquad -P_N((4N+2)(x-n))\}\\
&=\operatorname{sgn}(\cos(4N+2)\,\pi x)\sum_{n=-\infty}^{\infty}\{V_N((4N+2)(x-n))\\
&\qquad -P_N((4N+2)(x-n))\}\\
&=\operatorname{sgn}(\cos(4N+2)\,\pi x)\{u(x)-p_N(x)\}.
\end{aligned}$$

Next, using (1.28) and (3.16) we have

$$\int_{-1/2}^{1/2} |u(x)-p_N(x)|\,dx$$

$$=\int_{-1/2}^{1/2} \operatorname{sgn}(\cos(4N+2)\,\pi x)\{u(x)-p_N(x)\}\,dx$$

$$=\sum_{n=-\infty}^{\infty}\int_{-1/2}^{1/2} \operatorname{sgn}(\cos(4N+2)\,\pi x)\{V_N((4N+2)(x-n))$$

$$-P_N((4N+2)(x-n))\}\,dx$$

$$=(4N+2)^{-1}\int_{-\infty}^{\infty} \operatorname{sgn}(\cos \pi x)\{V_N(x)-P_N(x)\}\,dx.$$

$$=(4N+2)^{-1}\int_{-\infty}^{\infty} |V_N(x)-P_N(x)|\,dx=(4N+2)^{-2}.$$

Thus (1.29) holds. Finally, we note that $\operatorname{sgn}(\cos(4N+2)\,\pi x)$ is periodic and has Fourier series expansion

$$\operatorname{sgn}(\cos(4N+2)\,\pi x)=\frac{2}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\,(2n+1)^{-1}\,e((2N+1)(2n+1)x). \tag{3.22}$$

If $f(x)$ is any trigonometric polynomial of degree $2N$ or less, then it follows from (3.22) that

$$\int_{-1/2}^{1/2} f(x)\operatorname{sgn}(\cos(4N+2)\,\pi x)\,dx=0. \tag{3.23}$$

Therefore for such a polynomial we have

$$\int_{-1/2}^{1/2} |u(x)-f(x)|\,dx \geqslant \left|\int_{-1/2}^{1/2} \operatorname{sgn}(\cos(4N+2)\,\pi x)\{u(x)-f(x)\}\,dx\right|$$

$$=\left|\int_{-1/2}^{1/2} (1-2\,|x|)\,\frac{2}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\,(2n+1)^{-1}\right.$$

$$\left.\times\, e((2N+1)(2n+1)x)\,dx\right|$$

$$= \frac{4}{\pi} \left| \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^{-1} \right.$$
$$\left. \times \int_{-1/2}^{1/2} |x| \, e((2N+1)(2n+1)x) \, dx \right|$$
$$= \frac{4}{\pi^3} (2N+1)^{-2} \left| \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^{-3} \right|$$
$$= (4N+2)^{-2}.$$

This establishes the lower bound given in (1.30). In order for equality to occur, it must be that $f(x) = u(x)$ at each zero of $\cos(4N+2)\pi x$. There are $4N+1$ distinct roots (mod 1), and these conditions completely determine a trigonometric polynomial of degree $2N$ or less ([13], Vol. II, pp. 1–3). Therefore $p_N(x)$ is unique.

4. PROOFS OF REMAINING RESULTS

Proof of Corollary 1. To begin, if $0 < \alpha \leqslant 1/2$, then we may take $B_\alpha(z) = B(z)$ which is Beurling's function defined in (1.1). From (1.3) we see that (1.33) holds in this case, and an application of Theorem 1 in the case of $B_{1/2}(z)$ shows that the inequality (1.32) is valid as well. Now suppose that $\alpha > 1/2$, and then define

$$\tilde{\alpha} = [\alpha + 1/2] - 1/2.$$

Note that if $\alpha = N + 1/2$, then $\tilde{\alpha} = \alpha$, and in general we have $\tilde{\alpha} \leqslant \alpha$. It is clear that $\tilde{\alpha} = N + 1/2$ for some integer $N \geqslant 0$, so that $B_{\tilde{\alpha}}(z)$ is given in (1.7). Therefore we may now define

$$B_\alpha(z) = B_{\tilde{\alpha}}(\tilde{\alpha}\alpha^{-1} z).$$

It is clear that $B_\alpha(z)$ is entire of exponential type at most 2π. Furthermore we have

$$R_\alpha(x) = R_{\tilde{\alpha}}(\tilde{\alpha}\alpha^{-1} x)$$
$$\leqslant B_{\tilde{\alpha}}(\tilde{\alpha}\alpha^{-1} x) = B_\alpha(x)$$

for all real x, so that (1.32) holds. Finally,

$$\int_{-\infty}^{\infty} B_\alpha(x) - R_\alpha(x)\,dx = \alpha\tilde{\alpha}^{-1} \int_{-\infty}^{\infty} B_{\tilde{\alpha}}(x) - R_{\tilde{\alpha}}(x)\,dx$$
$$= (\alpha\tilde{\alpha}^{-1})(4\tilde{\alpha})^{-1}$$
$$= \Delta(\alpha),$$

which verifies (1.33) and completes the proof.

Proof of Theorem 5. If β is any real number and $F(z)$ is an integrable entire function of exponential type $2\pi\varepsilon$, then by the Paley–Wiener theorem it follows that

$$\int_{-\infty}^{\infty} T(x+\beta)\,F(x)\,dx = \int_{-\infty}^{\infty} \sum_{n=1}^{N} c_n e(\lambda_n(x+\beta))\,F(x)\,dx$$
$$= \sum_{n=1}^{N} c_n e(\lambda_n \beta) \int_{-\infty}^{\infty} F(x)\,e(\lambda_n x)\,dx$$
$$= \sum_{n=1}^{N} c_n e(\lambda_n \beta)\,\hat{F}(-\lambda_n) = 0.$$

Therefore we have

$$s^{-1} \int_{r}^{r+s} T(x)\,dx = \tfrac{1}{2} \int_{-\infty}^{\infty} T(x+r+s/2)\,dS_{s/2}(x)$$
$$= \tfrac{1}{2} \int_{-\infty}^{\infty} T(x+r+s/2)\,d\{S_{\varepsilon s}(2\varepsilon x) - G_{\varepsilon s}(2\varepsilon x)\}$$
$$= -\tfrac{1}{2} \int_{-\infty}^{\infty} \{S_{\varepsilon s}(2\varepsilon x)\} - G_{\varepsilon s}(2\varepsilon x)\}\;T'(x+r+s/2)\,dx.$$

Thus by Corollary 2,

$$\left| s^{-1} \int_{r}^{r+s} T(x)\,dx \right| \leqslant (\sup_t |T'(t)|) \left(\tfrac{1}{2} \int_{-\infty}^{\infty} |S_{\varepsilon s}(2\varepsilon x) - G_{\varepsilon s}(2\varepsilon x)|\,dx \right)$$
$$= (4\varepsilon)^{-1}\,\Delta(\varepsilon s) \sup_t |T'(t)|,$$

which verifies (1.36). If the additional assumption that $T(x)$ is real-valued is included, then

$$s^{-1}\int_r^{r+s} T(x)\,dx = \tfrac{1}{2}\int_{-\infty}^{\infty} T(x+r+s)\,dR_s(x)$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} T(x+r+s)\,d\{R_{\varepsilon s}(\varepsilon x) - B_{\varepsilon s}(\varepsilon x)\}$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} \{B_{\varepsilon s}(\varepsilon x) - R_{\varepsilon s}(\varepsilon x)\}\; T'(x+r+s)\,dx$$

$$\leqslant \tfrac{1}{2}\,(\sup_t T'(t)) \int_{-\infty}^{\infty} B_{\varepsilon s}(\varepsilon x)\,dx - R_{\varepsilon s}(\varepsilon x)\,dx$$

$$= (2\varepsilon)^{-1}\,\Delta(\varepsilon s)\,\sup_t T'(t).$$

Using $b_{\varepsilon s}(\varepsilon x)$ in place of $B_{\varepsilon s}(\varepsilon x)$ above and proceeding in the same manner, we have

$$s^{-1}\int_r^{r+s} T(x)\,dx \geqslant -(2\varepsilon)^{-1}\,\Delta(\varepsilon s)\,\sup_t T'(t),$$

so that (1.37) holds and the proof is complete.

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